

YEAR 11 TO YEAR 12 TRANSITION BOOKLET

*Preparation for A-Level Mathematics
(Extra Question Bank)*

Bridging the Gap to Key Stage 5

This booklet contains a structured collection of supplementary practice questions focusing on core algebraic manipulation. Mastering these extension topics is essential for establishing fluency and maintaining a confident start to your A-Level Mathematics course.



CONTENTS

Y11 to Y12 Bridging Work

Topics

Expanding brackets and simplifying expressions	1
Surds and rationalising the denominator	4
Rules of indices	9
Factorising expressions	14
Completing the square.....	18
Solving quadratic equations by factorisation	21
Solving quadratic equations by completing the square	23
Solving quadratic equations by using the formula	25
Solving linear simultaneous equations using the elimination method.....	28
Solving linear simultaneous equations using the substitution method.....	30
Solving linear and quadratic simultaneous equations	33
Linear inequalities.....	36
Quadratic inequalities	39
Rearranging equations	42

Expanding brackets and simplifying expressions

A LEVEL LINKS

Scheme of work: 1a. Algebraic expressions – basic algebraic manipulation, indices and surds

Key points

- When you expand one set of brackets you must multiply everything inside the bracket by what is outside.
- When you expand two linear expressions, each with two terms of the form $ax + b$, where $a \neq 0$ and $b \neq 0$, you create four terms. Two of these can usually be simplified by collecting like terms.

Examples

Example 1 Expand $4(3x - 2)$

$4(3x - 2) = 12x - 8$	Multiply everything inside the bracket by the 4 outside the bracket
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Example 2 Expand and simplify $3(x + 5) - 4(2x + 3)$

$3(x + 5) - 4(2x + 3)$ $= 3x + 15 - 8x - 12$ $= 3 - 5x$	1 Expand each set of brackets separately by multiplying $(x + 5)$ by 3 and $(2x + 3)$ by -4 2 Simplify by collecting like terms: $3x - 8x = -5x$ and $15 - 12 = 3$
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Example 3 Expand and simplify $(x + 3)(x + 2)$

$(x + 3)(x + 2)$ $= x(x + 2) + 3(x + 2)$ $= x^2 + 2x + 3x + 6$ $= x^2 + 5x + 6$	1 Expand the brackets by multiplying $(x + 2)$ by x and $(x + 2)$ by 3 2 Simplify by collecting like terms: $2x + 3x = 5x$
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Example 4 Expand and simplify $(x - 5)(2x + 3)$

$(x - 5)(2x + 3)$ $= x(2x + 3) - 5(2x + 3)$ $= 2x^2 + 3x - 10x - 15$ $= 2x^2 - 7x - 15$	1 Expand the brackets by multiplying $(2x + 3)$ by x and $(2x + 3)$ by -5 2 Simplify by collecting like terms: $3x - 10x = -7x$
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Practice

1 Expand.

a $3(2x - 1)$

c $-(3xy - 2y^2)$

b $-2(5pq + 4q^2)$

2 Expand and simplify.

a $7(3x + 5) + 6(2x - 8)$

c $9(3s + 1) - 5(6s - 10)$

b $8(5p - 2) - 3(4p + 9)$

d $2(4x - 3) - (3x + 5)$

3 Expand.

a $3x(4x + 8)$

c $-2h(6h^2 + 11h - 5)$

b $4k(5k^2 - 12)$

d $-3s(4s^2 - 7s + 2)$

4 Expand and simplify.

a $3(y^2 - 8) - 4(y^2 - 5)$

c $4p(2p - 1) - 3p(5p - 2)$

b $2x(x + 5) + 3x(x - 7)$

d $3b(4b - 3) - b(6b - 9)$

5 Expand $\frac{1}{2}(2y - 8)$

6 Expand and simplify.

a $13 - 2(m + 7)$

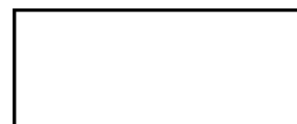
b $5p(p^2 + 6p) - 9p(2p - 3)$

7 The diagram shows a rectangle.

Write down an expression, in terms of x , for the area of the rectangle.

Show that the area of the rectangle can be written as $21x^2 - 35x$

$3x - 5$



$7x$

8 Expand and simplify.

a $(x + 4)(x + 5)$

c $(x + 7)(x - 2)$

e $(2x + 3)(x - 1)$

g $(5x - 3)(2x - 5)$

i $(3x + 4y)(5y + 6x)$

k $(2x - 7)^2$

b $(x + 7)(x + 3)$

d $(x + 5)(x - 5)$

f $(3x - 2)(2x + 1)$

h $(3x - 2)(7 + 4x)$

j $(x + 5)^2$

l $(4x - 3y)^2$

Extend

9 Expand and simplify $(x + 3)^2 + (x - 4)^2$

10 Expand and simplify.

a $\left(x + \frac{1}{x}\right)\left(x - \frac{2}{x}\right)$

b $\left(x + \frac{1}{x}\right)^2$

Watch out!

When multiplying (or dividing) positive and negative numbers, if the signs are the same the answer is '+'; if the signs are different the answer is '-'.

Answers

1 **a** $6x - 3$
 c $-3xy + 2y^2$

b $-10pq - 8q^2$

2 a $21x + 35 + 12x - 48 = 33x - 13$

b $40p - 16 - 12p - 27 = 28p - 43$

c $27s + 9 - 30s + 50 = -3s + 59 = 59 - 3s$

d $8x - 6 - 3x - 5 = 5x - 11$

3 a $12x^2 + 24x$

b $20k^3 - 48k$

c $10h - 12h^3 - 22h^2$

d $21s^2 - 21s^3 - 6s$

4 a $-y^2 - 4$

b $5x^2 - 11x$

c $2p - 7p^2$

d $6b^2$

5 $y - 4$

6 a $-1 - 2m$

b $5p^3 + 12p^2 + 27p$

7 $7x(3x - 5) = 21x^2 - 35x$

8 a $x^2 + 9x + 20$

b $x^2 + 10x + 21$

c $x^2 + 5x - 14$

d $x^2 - 25$

e $2x^2 + x - 3$

f $6x^2 - x - 2$

g $10x^2 - 31x + 15$

h $12x^2 + 13x - 14$

i $18x^2 + 39xy + 20y^2$

j $x^2 + 10x + 25$

k $4x^2 - 28x + 49$

$$16x^2 - 24xy + 9y^2$$

9 $2x^2 - 2x + 25$

10 a $x^2 - 1 - \frac{2}{x^2}$

b $x^2 + 2 + \frac{1}{x^2}$

Surds and rationalising the denominator

A LEVEL LINKS

Scheme of work: 1a. Algebraic expressions – basic algebraic manipulation, indices and surds

Key points

- A surd is the square root of a number that is not a square number, for example $\sqrt{2}, \sqrt{3}, \sqrt{5}$, etc.
- Surds can be used to give the exact value for an answer.
- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
- To rationalise the denominator means to remove the surd from the denominator of a fraction.
- To rationalise $\frac{a}{\sqrt{b}}$ you multiply the numerator and denominator by the surd $\sqrt{b}$
- To rationalise $\frac{a}{b + \sqrt{c}}$ you multiply the numerator and denominator by $b - \sqrt{c}$

Examples

Example 1 Simplify $\sqrt{50}$

$\begin{aligned}\sqrt{50} &= \sqrt{25 \times 2} \\ &= \sqrt{25} \times \sqrt{2} \\ &= 5 \times \sqrt{2} \\ &= 5\sqrt{2}\end{aligned}$	1 Choose two numbers that are factors of 50. One of the factors must be a square number 2 Use the rule $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ 3 Use $\sqrt{25} = 5$
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Example 2 Simplify $\sqrt{147} - 2\sqrt{12}$

$\begin{aligned}\sqrt{147} - 2\sqrt{12} \\ &= \sqrt{49 \times 3} - 2\sqrt{4 \times 3} \\ &= \sqrt{49} \times \sqrt{3} - 2\sqrt{4} \times \sqrt{3} \\ &= 7 \times \sqrt{3} - 2 \times 2 \times \sqrt{3} \\ &= 7\sqrt{3} - 4\sqrt{3} \\ &= 3\sqrt{3}\end{aligned}$	1 Simplify $\sqrt{147}$ and $2\sqrt{12}$. Choose two numbers that are factors of 147 and two numbers that are factors of 12. One of each pair of factors must be a square number 2 Use the rule $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ 3 Use $\sqrt{49} = 7$ and $\sqrt{4} = 2$ 4 Collect like terms
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Example 3 Simplify $(\sqrt{7} + \sqrt{2})(\sqrt{7} - \sqrt{2})$

$ \begin{aligned} &(\sqrt{7} + \sqrt{2})(\sqrt{7} - \sqrt{2}) \\ &= \sqrt{49} - \sqrt{7}\sqrt{2} + \sqrt{2}\sqrt{7} - \sqrt{4} \\ &= 7 - 2 \\ &= 5 \end{aligned} $	1 Expand the brackets. A common mistake here is to write $(\sqrt{7})^2 = 49$ 2 Collect like terms: $-\sqrt{7}\sqrt{2} + \sqrt{2}\sqrt{7} = -\sqrt{7}\sqrt{2} + \sqrt{7}\sqrt{2} = 0$
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Example 4 Rationalise $\frac{1}{\sqrt{3}}$

$ \begin{aligned} \frac{1}{\sqrt{3}} &= \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{1 \times \sqrt{3}}{\sqrt{9}} \\ &= \frac{\sqrt{3}}{3} \end{aligned} $	1 Multiply the numerator and denominator by $\sqrt{3}$ 2 Use $\sqrt{9} = 3$
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Example 5 Rationalise and simplify $\frac{\sqrt{2}}{\sqrt{12}}$

$ \begin{aligned} \frac{\sqrt{2}}{\sqrt{12}} &= \frac{\sqrt{2}}{\sqrt{12}} \times \frac{\sqrt{12}}{\sqrt{12}} \\ &= \frac{\sqrt{2} \times \sqrt{4 \times 3}}{12} \\ &= \frac{2\sqrt{2}\sqrt{3}}{12} \\ &= \frac{\sqrt{2}\sqrt{3}}{6} \end{aligned} $	1 Multiply the numerator and denominator by $\sqrt{12}$ 2 Simplify $\sqrt{12}$ in the numerator. Choose two numbers that are factors of 12. One of the factors must be a square number 3 Use the rule $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ 4 Use $\sqrt{4} = 2$ 5 Simplify the fraction: $\frac{2}{12}$ simplifies to $\frac{1}{6}$
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Example 6 Rationalise and simplify $\frac{3}{2+\sqrt{5}}$

$\frac{3}{2+\sqrt{5}} = \frac{3}{2+\sqrt{5}} \times \frac{2-\sqrt{5}}{2-\sqrt{5}}$ $= \frac{3(2-\sqrt{5})}{(2+\sqrt{5})(2-\sqrt{5})}$ $= \frac{6-3\sqrt{5}}{4+2\sqrt{5}-2\sqrt{5}-5}$ $= \frac{6-3\sqrt{5}}{-1}$ $= 3\sqrt{5}-6$	1 Multiply the numerator and denominator by $2-\sqrt{5}$ 2 Expand the brackets 3 Simplify the fraction 4 Divide the numerator by -1 Remember to change the sign of all terms when dividing by -1
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Practice

1 Simplify.

- a** $\sqrt{45}$
c $\sqrt{48}$
e $\sqrt{300}$
g $\sqrt{72}$

- b** $\sqrt{125}$
d $\sqrt{175}$
f $\sqrt{28}$
h $\sqrt{162}$

Hint

One of the two numbers you choose at the start must be a square number.

2 Simplify.

- a** $\sqrt{72} + \sqrt{162}$
c $\sqrt{50} - \sqrt{8}$
e $2\sqrt{28} + \sqrt{28}$

- b** $\sqrt{45} - 2\sqrt{5}$
d $\sqrt{75} - \sqrt{48}$
f $2\sqrt{12} - \sqrt{12} + \sqrt{27}$

Watch out!

Check you have chosen the highest square number at the start.

3 Expand and simplify.

- a** $(\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3})$
c $(4 - \sqrt{5})(\sqrt{45} + 2)$

- b** $(3 + \sqrt{3})(5 - \sqrt{12})$
d $(5 + \sqrt{2})(6 - \sqrt{8})$

4 Rationalise and simplify, if possible.

a $\frac{1}{\sqrt{5}}$

b $\frac{1}{\sqrt{11}}$

c $\frac{2}{\sqrt{7}}$

d $\frac{2}{\sqrt{8}}$

e $\frac{2}{\sqrt{2}}$

f $\frac{5}{\sqrt{5}}$

g $\frac{\sqrt{8}}{\sqrt{24}}$

h $\frac{\sqrt{5}}{\sqrt{45}}$

5 Rationalise and simplify.

a $\frac{1}{3-\sqrt{5}}$

b $\frac{2}{4+\sqrt{3}}$

c $\frac{6}{5-\sqrt{2}}$

Extend

6 Expand and simplify $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})$

7 Rationalise and simplify, if possible.

a $\frac{1}{\sqrt{9} - \sqrt{8}}$

b $\frac{1}{\sqrt{x} - \sqrt{y}}$

Answers

1 **a** $3\sqrt{5}$
 c $4\sqrt{3}$
 e $10\sqrt{3}$
 g $6\sqrt{2}$

b $5\sqrt{5}$
 d $5\sqrt{7}$
 f $2\sqrt{7}$
 h $9\sqrt{2}$

2 **a** $15\sqrt{2}$
 c $3\sqrt{2}$
 e $6\sqrt{7}$

b $\sqrt{5}$
 d $\sqrt{3}$
 f $5\sqrt{3}$

3 **a** -1
 c $10\sqrt{5}-7$

b $9-\sqrt{3}$
 d $26-4\sqrt{2}$

4 **a** $\frac{\sqrt{5}}{5}$
 c $\frac{2\sqrt{7}}{7}$
 e $\sqrt{2}$
 g $\frac{\sqrt{3}}{3}$

b $\frac{\sqrt{11}}{11}$
 d $\frac{\sqrt{2}}{2}$
 f $\sqrt{5}$
 h $\frac{1}{3}$

5 **a** $\frac{3+\sqrt{5}}{4}$

b $\frac{2(4-\sqrt{3})}{13}$

c $\frac{6(5+\sqrt{2})}{23}$

6 $x-y$

7 **a** $3+2\sqrt{2}$

b $\frac{\sqrt{x}+\sqrt{y}}{x-y}$

Rules of indices

A LEVEL LINKS

Scheme of work: 1a. Algebraic expressions – basic algebraic manipulation, indices and surds

Key points

- $a^m \times a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $a^0 = 1$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$ i.e. the n th root of a
- $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$
- $a^{-m} = \frac{1}{a^m}$
- The square root of a number produces two solutions, e.g. $\sqrt{16} = \pm 4$.

Examples

Example 1 Evaluate 10^0

$10^0 = 1$	Any value raised to the power of zero is equal to 1
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Example 2 Evaluate $9^{\frac{1}{2}}$

$9^{\frac{1}{2}} = \sqrt{9}$ $= 3$	Use the rule $a^{\frac{1}{n}} = \sqrt[n]{a}$
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Example 3 Evaluate $27^{\frac{2}{3}}$

$27^{\frac{2}{3}} = \left(\sqrt[3]{27}\right)^2$ $= 3^2$ $= 9$	1 Use the rule $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$ 2 Use $\sqrt[3]{27} = 3$
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Example 4 Evaluate 4^{-2}

$4^{-2} = \frac{1}{4^2}$ $= \frac{1}{16}$	1 Use the rule $a^{-m} = \frac{1}{a^m}$ 2 Use $4^2 = 16$
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Example 5 Simplify $\frac{6x^5}{2x^2}$

$\frac{6x^5}{2x^2} = 3x^3$	$6 \div 2 = 3$ and use the rule $\frac{a^m}{a^n} = a^{m-n}$ to give $\frac{x^5}{x^2} = x^{5-2} = x^3$
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Example 6 Simplify $\frac{x^3 \times x^5}{x^4}$

$\frac{x^3 \times x^5}{x^4} = \frac{x^{3+5}}{x^4} = \frac{x^8}{x^4}$ $= x^{8-4} = x^4$	1 Use the rule $a^m \times a^n = a^{m+n}$ 2 Use the rule $\frac{a^m}{a^n} = a^{m-n}$
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Example 7 Write $\frac{1}{3x}$ as a single power of x

$\frac{1}{3x} = \frac{1}{3}x^{-1}$	Use the rule $\frac{1}{a^m} = a^{-m}$, note that the fraction $\frac{1}{3}$ remains unchanged
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Example 8 Write $\frac{4}{\sqrt{x}}$ as a single power of x

$\frac{4}{\sqrt{x}} = \frac{4}{x^{\frac{1}{2}}}$ $= 4x^{-\frac{1}{2}}$	1 Use the rule $a^{\frac{1}{n}} = \sqrt[n]{a}$ 2 Use the rule $\frac{1}{a^m} = a^{-m}$
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Practice

1 Evaluate.

a 14^0

b 3^0

c 5^0

d x^0

2 Evaluate.

a $49^{\frac{1}{2}}$

b $64^{\frac{1}{3}}$

c $125^{\frac{1}{3}}$

d $16^{\frac{1}{4}}$

3 Evaluate.

a $25^{\frac{3}{2}}$

b $8^{\frac{5}{3}}$

c $49^{\frac{3}{2}}$

d $16^{\frac{3}{4}}$

4 Evaluate.

a 5^{-2}

b 4^{-3}

c 2^{-5}

d 6^{-2}

5 Simplify.

a $\frac{3x^2 \times x^3}{2x^2}$

b $\frac{10x^5}{2x^2 \times x}$

c $\frac{3x \times 2x^3}{2x^3}$

d $\frac{7x^3y^2}{14x^5y}$

e $\frac{y^2}{y^{\frac{1}{2}} \times y}$

f $\frac{c^{\frac{1}{2}}}{c^2 \times c^{\frac{3}{2}}}$

g $\frac{(2x^2)^3}{4x^0}$

h $\frac{x^{\frac{1}{2}} \times x^{\frac{3}{2}}}{x^{-2} \times x^3}$

Watch out!

Remember that any value raised to the power of zero is 1. This is the rule $a^0 = 1$.

6 Evaluate.

a $4^{-\frac{1}{2}}$

b $27^{-\frac{2}{3}}$

c $9^{-\frac{1}{2}} \times 2^3$

d $16^{\frac{1}{4}} \times 2^{-3}$

e $\left(\frac{9}{16}\right)^{-\frac{1}{2}}$

f $\left(\frac{27}{64}\right)^{-\frac{2}{3}}$

7 Write the following as a single power of x .

a $\frac{1}{x}$

b $\frac{1}{x^7}$

c $\sqrt[4]{x}$

d $\sqrt[5]{x^2}$

e $\frac{1}{\sqrt[3]{x}}$

f $\frac{1}{\sqrt[3]{x^2}}$

8 Write the following without negative or fractional powers.

a x^{-3}

b x^0

c $x^{\frac{1}{5}}$

d $x^{\frac{2}{5}}$

e $x^{-\frac{1}{2}}$

f $x^{-\frac{3}{4}}$

9 Write the following in the form ax^n .

a $5\sqrt{x}$

b $\frac{2}{x^3}$

c $\frac{1}{3x^4}$

d $\frac{2}{\sqrt{x}}$

e $\frac{4}{\sqrt[3]{x}}$

f 3

Extend

10 Write as sums of powers of x .

a $\frac{x^5 + 1}{x^2}$

b $x^2\left(x + \frac{1}{x}\right)$

c $x^{-4}\left(x^2 + \frac{1}{x^3}\right)$

Answers

- | | | | | | | | | |
|-----------|----------|---------------------|----------|----------------------|----------|---------------------------|----------|----------------|
| 1 | a | 1 | b | 1 | c | 1 | d | 1 |
| 2 | a | 7 | b | 4 | c | 5 | d | 2 |
| 3 | a | 125 | b | 32 | c | 343 | d | 8 |
| 4 | a | $\frac{1}{25}$ | b | $\frac{1}{64}$ | c | $\frac{1}{32}$ | d | $\frac{1}{36}$ |
| 5 | a | $\frac{3x^3}{2}$ | b | $5x^2$ | | | | |
| | c | $3x$ | d | $\frac{y}{2x^2}$ | | | | |
| | e | $y^{\frac{1}{2}}$ | f | c^{-3} | | | | |
| | g | $2x^6$ | h | x | | | | |
| 6 | a | $\frac{1}{2}$ | b | $\frac{1}{9}$ | c | $\frac{8}{3}$ | | |
| | d | $\frac{1}{4}$ | e | $\frac{4}{3}$ | f | $\frac{16}{9}$ | | |
| 7 | a | x^{-1} | b | x^{-7} | c | $x^{\frac{1}{4}}$ | | |
| | d | $x^{\frac{2}{5}}$ | e | $x^{-\frac{1}{3}}$ | f | $x^{-\frac{2}{3}}$ | | |
| 8 | a | $\frac{1}{x^3}$ | b | 1 | c | $\sqrt[5]{x}$ | | |
| | d | $\sqrt[5]{x^2}$ | e | $\frac{1}{\sqrt{x}}$ | f | $\frac{1}{\sqrt[4]{x^3}}$ | | |
| 9 | a | $5x^{\frac{1}{2}}$ | b | $2x^{-3}$ | c | $\frac{1}{3}x^{-4}$ | | |
| | d | $2x^{-\frac{1}{2}}$ | e | $4x^{-\frac{1}{3}}$ | f | $3x^0$ | | |
| 10 | a | $x^3 + x^{-2}$ | b | $x^3 + x$ | c | $x^{-2} + x^{-7}$ | | |

Factorising expressions

A LEVEL LINKS

Scheme of work: 1b. Quadratic functions – factorising, solving, graphs and the discriminants

Key points

- Factorising an expression is the opposite of expanding the brackets.
- A quadratic expression is in the form $ax^2 + bx + c$, where $a \neq 0$.
- To factorise a quadratic equation find two numbers whose sum is b and whose product is ac .
- An expression in the form $x^2 - y^2$ is called the difference of two squares. It factorises to $(x - y)(x + y)$.

Examples

Example 1 Factorise $15x^2y^3 + 9x^4y$

$15x^2y^3 + 9x^4y = 3x^2y(5y^2 + 3x^2)$	The highest common factor is $3x^2y$. So take $3x^2y$ outside the brackets and then divide each term by $3x^2y$ to find the terms in the brackets
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Example 2 Factorise $4x^2 - 25y^2$

$4x^2 - 25y^2 = (2x + 5y)(2x - 5y)$	This is the difference of two squares as the two terms can be written as $(2x)^2$ and $(5y)^2$
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Example 3 Factorise $x^2 + 3x - 10$

$b = 3, ac = -10$ So $x^2 + 3x - 10 = x^2 + 5x - 2x - 10$ $= x(x + 5) - 2(x + 5)$ $= (x + 5)(x - 2)$	1 Work out the two factors of $ac = -10$ which add to give $b = 3$ (5 and -2) 2 Rewrite the b term ($3x$) using these two factors 3 Factorise the first two terms and the last two terms 4 $(x + 5)$ is a factor of both terms
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Example 4 Factorise $6x^2 - 11x - 10$

$b = -11, ac = -60$ So $6x^2 - 11x - 10 = 6x^2 - 15x + 4x - 10$ $= 3x(2x - 5) + 2(2x - 5)$ $= (2x - 5)(3x + 2)$	1 Work out the two factors of $ac = -60$ which add to give $b = -11$ (-15 and 4) 2 Rewrite the b term ($-11x$) using these two factors 3 Factorise the first two terms and the last two terms 4 $(2x - 5)$ is a factor of both terms
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Example 5 Simplify $\frac{x^2 - 4x - 21}{2x^2 + 9x + 9}$

$\frac{x^2 - 4x - 21}{2x^2 + 9x + 9}$ For the numerator: $b = -4, ac = -21$ So $x^2 - 4x - 21 = x^2 - 7x + 3x - 21$ $= x(x - 7) + 3(x - 7)$ $= (x - 7)(x + 3)$ For the denominator: $b = 9, ac = 18$ So $2x^2 + 9x + 9 = 2x^2 + 6x + 3x + 9$ $= 2x(x + 3) + 3(x + 3)$ $= (x + 3)(2x + 3)$ So $\frac{x^2 - 4x - 21}{2x^2 + 9x + 9} = \frac{(x - 7)(x + 3)}{(x + 3)(2x + 3)}$ $= \frac{x - 7}{2x + 3}$	1 Factorise the numerator and the denominator 2 Work out the two factors of $ac = -21$ which add to give $b = -4$ (-7 and 3) 3 Rewrite the b term ($-4x$) using these two factors 4 Factorise the first two terms and the last two terms 5 $(x - 7)$ is a factor of both terms 6 Work out the two factors of $ac = 18$ which add to give $b = 9$ (6 and 3) 7 Rewrite the b term ($9x$) using these two factors 8 Factorise the first two terms and the last two terms 9 $(x + 3)$ is a factor of both terms 10 $(x + 3)$ is a factor of both the numerator and denominator so cancels out as a value divided by itself is 1
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Practice

1 Factorise.

a $6x^4y^3 - 10x^3y^4$

c $25x^2y^2 - 10x^3y^2 + 15x^2y^3$

b $21a^3b^5 + 35a^5b^2$

2 Factorise

a $x^2 + 7x + 12$

c $x^2 - 11x + 30$

e $x^2 - 7x - 18$

g $x^2 - 3x - 40$

b $x^2 + 5x - 14$

d $x^2 - 5x - 24$

f $x^2 + x - 20$

h $x^2 + 3x - 28$

3 Factorise

a $36x^2 - 49y^2$

c $18a^2 - 200b^2c^2$

b $4x^2 - 81y^2$

4 Factorise

a $2x^2 + x - 3$

c $2x^2 + 7x + 3$

e $10x^2 + 21x + 9$

b $6x^2 + 17x + 5$

d $9x^2 - 15x + 4$

f $12x^2 - 38x + 20$

5 Simplify the algebraic fractions.

a $\frac{2x^2 + 4x}{x^2 - x}$

c $\frac{x^2 - 2x - 8}{x^2 - 4x}$

e $\frac{x^2 - x - 12}{x^2 - 4x}$

b $\frac{x^2 + 3x}{x^2 + 2x - 3}$

d $\frac{x^2 - 5x}{x^2 - 25}$

f $\frac{2x^2 + 14x}{2x^2 + 4x - 70}$

6 Simplify

a $\frac{9x^2 - 16}{3x^2 + 17x - 28}$

c $\frac{4 - 25x^2}{10x^2 - 11x - 6}$

b $\frac{2x^2 - 7x - 15}{3x^2 - 17x + 10}$

d $\frac{6x^2 - x - 1}{2x^2 + 7x - 4}$

Hint

Take the highest common factor outside the bracket.

Extend

7 Simplify $\sqrt{x^2 + 10x + 25}$

8 Simplify $\frac{(x+2)^2 + 3(x+2)^2}{x^2 - 4}$

Answers

- 1** **a** $2x^3y^3(3x - 5y)$ **b** $7a^3b^2(3b^3 + 5a^2)$
 c $5x^2y^2(5 - 2x + 3y)$
- 2** **a** $(x + 3)(x + 4)$ **b** $(x + 7)(x - 2)$
 c $(x - 5)(x - 6)$ **d** $(x - 8)(x + 3)$
 e $(x - 9)(x + 2)$ **f** $(x + 5)(x - 4)$
 g $(x - 8)(x + 5)$ **h** $(x + 7)(x - 4)$
- 3** **a** $(6x - 7y)(6x + 7y)$ **b** $(2x - 9y)(2x + 9y)$
 c $2(3a - 10bc)(3a + 10bc)$
- 4** **a** $(x - 1)(2x + 3)$ **b** $(3x + 1)(2x + 5)$
 c $(2x + 1)(x + 3)$ **d** $(3x - 1)(3x - 4)$
 e $(5x + 3)(2x + 3)$ **f** $2(3x - 2)(2x - 5)$
- 5** **a** $\frac{2(x+2)}{x-1}$ **b** $\frac{x}{x-1}$
 c $\frac{x+2}{x}$ **d** $\frac{x}{x+5}$
 e $\frac{x+3}{x}$ **f** $\frac{x}{x-5}$
- 6** **a** $\frac{3x+4}{x+7}$ **b** $\frac{2x+3}{3x-2}$
 c $\frac{2-5x}{2x-3}$ **d** $\frac{3x+1}{x+4}$
- 7** $(x + 5)$
- 8** $\frac{4(x+2)}{x-2}$

Completing the square

A LEVEL LINKS

Scheme of work: 1b. Quadratic functions – factorising, solving, graphs and the discriminants

Key points

- Completing the square for a quadratic rearranges $ax^2 + bx + c$ into the form $p(x + q)^2 + r$
- If $a \neq 1$, then factorise using a as a common factor.

Examples

Example 1 Complete the square for the quadratic expression $x^2 + 6x - 2$

$x^2 + 6x - 2$ $= (x + 3)^2 - 9 - 2$ $= (x + 3)^2 - 11$	1 Write $x^2 + bx + c$ in the form $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$ 2 Simplify
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Example 2 Write $2x^2 - 5x + 1$ in the form $p(x + q)^2 + r$

$2x^2 - 5x + 1$ $= 2\left(x^2 - \frac{5}{2}x\right) + 1$ $= 2\left[\left(x - \frac{5}{4}\right)^2 - \left(\frac{5}{4}\right)^2\right] + 1$ $= 2\left(x - \frac{5}{4}\right)^2 - \frac{25}{8} + 1$ $= 2\left(x - \frac{5}{4}\right)^2 - \frac{17}{8}$	1 Before completing the square write $ax^2 + bx + c$ in the form $a\left(x^2 + \frac{b}{a}x\right) + c$ 2 Now complete the square by writing $x^2 - \frac{5}{2}x$ in the form $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$ 3 Expand the square brackets – don't forget to multiply $\left(\frac{5}{4}\right)^2$ by the factor of 2 4 Simplify
--	---

Practice

- 1 Write the following quadratic expressions in the form $(x + p)^2 + q$
- | | |
|-------------------------|--------------------------|
| a $x^2 + 4x + 3$ | b $x^2 - 10x - 3$ |
| c $x^2 - 8x$ | d $x^2 + 6x$ |
| e $x^2 - 2x + 7$ | f $x^2 + 3x - 2$ |
- 2 Write the following quadratic expressions in the form $p(x + q)^2 + r$
- | | |
|---------------------------|---------------------------|
| a $2x^2 - 8x - 16$ | b $4x^2 - 8x - 16$ |
| c $3x^2 + 12x - 9$ | d $2x^2 + 6x - 8$ |
- 3 Complete the square.
- | | |
|--------------------------|--------------------------|
| a $2x^2 + 3x + 6$ | b $3x^2 - 2x$ |
| c $5x^2 + 3x$ | d $3x^2 + 5x + 3$ |

Extend

- 4 Write $(25x^2 + 30x + 12)$ in the form $(ax + b)^2 + c$.

Answers

1 a $(x+2)^2 - 1$

b $(x-5)^2 - 28$

c $(x-4)^2 - 16$

d $(x+3)^2 - 9$

e $(x-1)^2 + 6$

f $\left(x + \frac{3}{2}\right)^2 - \frac{17}{4}$

2 a $2(x-2)^2 - 24$

b $4(x-1)^2 - 20$

c $3(x+2)^2 - 21$

d $2\left(x + \frac{3}{2}\right)^2 - \frac{25}{2}$

3 a $2\left(x + \frac{3}{4}\right)^2 + \frac{39}{8}$

b $3\left(x - \frac{1}{3}\right)^2 - \frac{1}{3}$

c $5\left(x + \frac{3}{10}\right)^2 - \frac{9}{20}$

d $3\left(x + \frac{5}{6}\right)^2 + \frac{11}{12}$

4 $(5x+3)^2 + 3$

Solving quadratic equations by factorisation

A LEVEL LINKS

Scheme of work: 1b. Quadratic functions – factorising, solving, graphs and the discriminants

Key points

- A quadratic equation is an equation in the form $ax^2 + bx + c = 0$ where $a \neq 0$.
- To factorise a quadratic equation find two numbers whose sum is b and whose products is ac .
- When the product of two numbers is 0, then at least one of the numbers must be 0.
- If a quadratic can be solved it will have two solutions (these may be equal).

Examples

Example 1 Solve $5x^2 = 15x$

$5x^2 = 15x$ $5x^2 - 15x = 0$ $5x(x - 3) = 0$ So $5x = 0$ or $(x - 3) = 0$ Therefore $x = 0$ or $x = 3$	1 Rearrange the equation so that all of the terms are on one side of the equation and it is equal to zero. Do not divide both sides by x as this would lose the solution $x = 0$. 2 Factorise the quadratic equation. $5x$ is a common factor. 3 When two values multiply to make zero, at least one of the values must be zero. 4 Solve these two equations.
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Example 2 Solve $x^2 + 7x + 12 = 0$

$x^2 + 7x + 12 = 0$ $b = 7, ac = 12$ $x^2 + 4x + 3x + 12 = 0$ $x(x + 4) + 3(x + 4) = 0$ $(x + 4)(x + 3) = 0$ So $(x + 4) = 0$ or $(x + 3) = 0$ Therefore $x = -4$ or $x = -3$	1 Factorise the quadratic equation. Work out the two factors of $ac = 12$ which add to give you $b = 7$. (4 and 3) 2 Rewrite the b term ($7x$) using these two factors. 3 Factorise the first two terms and the last two terms. 4 $(x + 4)$ is a factor of both terms. 5 When two values multiply to make zero, at least one of the values must be zero. 6 Solve these two equations.
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Example 3 Solve $9x^2 - 16 = 0$

$9x^2 - 16 = 0$ $(3x + 4)(3x - 4) = 0$ So $(3x + 4) = 0$ or $(3x - 4) = 0$ $x = -\frac{4}{3}$ or $x = \frac{4}{3}$	- Factorise the quadratic equation. This is the difference of two squares as the two terms are $(3x)^2$ and $(4)^2$. - When two values multiply to make zero, at least one of the values must be zero. - Solve these two equations.
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Example 4 Solve $2x^2 - 5x - 12 = 0$

$b = -5, ac = -24$ So $2x^2 - 8x + 3x - 12 = 0$ $2x(x - 4) + 3(x - 4) = 0$ $(x - 4)(2x + 3) = 0$ So $(x - 4) = 0$ or $(2x + 3) = 0$ $x = 4$ or $x = -\frac{3}{2}$	- Factorise the quadratic equation. Work out the two factors of $ac = -24$ which add to give you $b = -5$. (-8 and 3) - Rewrite the b term ($-5x$) using these two factors. - Factorise the first two terms and the last two terms. $(x - 4)$ is a factor of both terms. - When two values multiply to make zero, at least one of the values must be zero. - Solve these two equations.
--	---

Practice

1 Solve

- | | |
|-------------------------------|--------------------------------|
| a $6x^2 + 4x = 0$ | b $28x^2 - 21x = 0$ |
| c $x^2 + 7x + 10 = 0$ | d $x^2 - 5x + 6 = 0$ |
| e $x^2 - 3x - 4 = 0$ | f $x^2 + 3x - 10 = 0$ |
| g $x^2 - 10x + 24 = 0$ | h $x^2 - 36 = 0$ |
| i $x^2 + 3x - 28 = 0$ | j $x^2 - 6x + 9 = 0$ |
| k $2x^2 - 7x - 4 = 0$ | l $3x^2 - 13x - 10 = 0$ |

2 Solve

- | | |
|---------------------------------|---------------------------------|
| a $x^2 - 3x = 10$ | b $x^2 - 3 = 2x$ |
| c $x^2 + 5x = 24$ | d $x^2 - 42 = x$ |
| e $x(x + 2) = 2x + 25$ | f $x^2 - 30 = 3x - 2$ |
| g $x(3x + 1) = x^2 + 15$ | h $3x(x - 1) = 2(x + 1)$ |

Hint

Get all terms onto one side of the equation.

Solving quadratic equations by completing the square

A LEVEL LINKS

Scheme of work: 1b. Quadratic functions – factorising, solving, graphs and the discriminants

Key points

- Completing the square lets you write a quadratic equation in the form $p(x + q)^2 + r = 0$.

Examples

Example 5 Solve $x^2 + 6x + 4 = 0$. Give your solutions in surd form.

$x^2 + 6x + 4 = 0$ $(x + 3)^2 - 9 + 4 = 0$ $(x + 3)^2 - 5 = 0$ $(x + 3)^2 = 5$ $x + 3 = \pm\sqrt{5}$ $x = \pm\sqrt{5} - 3$ $\text{So } x = -\sqrt{5} - 3 \text{ or } x = \sqrt{5} - 3$	- Write $x^2 + bx + c = 0$ in the form $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = 0$ - Simplify. - Rearrange the equation to work out x. First, add 5 to both sides. - Square root both sides. Remember that the square root of a value gives two answers. - Subtract 3 from both sides to solve the equation. - Write down both solutions.
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Example 6 Solve $2x^2 - 7x + 4 = 0$. Give your solutions in surd form.

$2x^2 - 7x + 4 = 0$ $2\left(x^2 - \frac{7}{2}x\right) + 4 = 0$ $2\left[\left(x - \frac{7}{4}\right)^2 - \left(\frac{7}{4}\right)^2\right] + 4 = 0$ $2\left(x - \frac{7}{4}\right)^2 - \frac{49}{8} + 4 = 0$ $2\left(x - \frac{7}{4}\right)^2 - \frac{17}{8} = 0$	- Before completing the square write $ax^2 + bx + c$ in the form $a\left(x^2 + \frac{b}{a}x\right) + c$ - Now complete the square by writing $x^2 - \frac{7}{2}x$ in the form $\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2$ - Expand the square brackets. - Simplify. <i>(continued on next page)</i>
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$2\left(x - \frac{7}{4}\right)^2 = \frac{17}{8}$ $\left(x - \frac{7}{4}\right)^2 = \frac{17}{16}$ $x - \frac{7}{4} = \pm \frac{\sqrt{17}}{4}$ $x = \pm \frac{\sqrt{17}}{4} + \frac{7}{4}$ $\text{So } x = \frac{7}{4} - \frac{\sqrt{17}}{4} \text{ or } x = \frac{7}{4} + \frac{\sqrt{17}}{4}$	5 Rearrange the equation to work out x. First, add $\frac{17}{8}$ to both sides. 6 Divide both sides by 2. 7 Square root both sides. Remember that the square root of a value gives two answers. 8 Add $\frac{7}{4}$ to both sides. 9 Write down both the solutions.
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Practice

3 Solve by completing the square.

a $x^2 - 4x - 3 = 0$

c $x^2 + 8x - 5 = 0$

e $2x^2 + 8x - 5 = 0$

b $x^2 - 10x + 4 = 0$

d $x^2 - 2x - 6 = 0$

f $5x^2 + 3x - 4 = 0$

4 Solve by completing the square.

a $(x - 4)(x + 2) = 5$

b $2x^2 + 6x - 7 = 0$

c $x^2 - 5x + 3 = 0$

Hint

Get all terms onto one side of the equation.

Solving quadratic equations by using the formula

A LEVEL LINKS

Scheme of work: 1b. Quadratic functions – factorising, solving, graphs and the discriminants

Key points

- Any quadratic equation of the form $ax^2 + bx + c = 0$ can be solved using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If $b^2 - 4ac$ is negative then the quadratic equation does not have any real solutions.
- It is useful to write down the formula before substituting the values for a , b and c .

Examples

Example 7 Solve $x^2 + 6x + 4 = 0$. Give your solutions in surd form.

$a = 1, b = 6, c = 4$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-6 \pm \sqrt{6^2 - 4(1)(4)}}{2(1)}$ $x = \frac{-6 \pm \sqrt{20}}{2}$ $x = \frac{-6 \pm 2\sqrt{5}}{2}$ $x = -3 \pm \sqrt{5}$ So $x = -3 - \sqrt{5}$ or $x = \sqrt{5} - 3$	- Identify a, b and c and write down the formula. Remember that $-b \pm \sqrt{b^2 - 4ac}$ is all over $2a$, not just part of it. - Substitute $a = 1$, $b = 6$, $c = 4$ into the formula. - Simplify. The denominator is 2, but this is only because $a = 1$. The denominator will not always be 2. - Simplify $\sqrt{20}$. $\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$ - Simplify by dividing numerator and denominator by 2. - Write down both the solutions.
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Example 8 Solve $3x^2 - 7x - 2 = 0$. Give your solutions in surd form.

$a = 3, b = -7, c = -2$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(-2)}}{2(3)}$ $x = \frac{7 \pm \sqrt{73}}{6}$ So $x = \frac{7 - \sqrt{73}}{6}$ or $x = \frac{7 + \sqrt{73}}{6}$	1 Identify a, b and c, making sure you get the signs right and write down the formula. Remember that $-b \pm \sqrt{b^2 - 4ac}$ is all over $2a$, not just part of it. 2 Substitute $a = 3$, $b = -7$, $c = -2$ into the formula. 3 Simplify. The denominator is 6 when $a = 3$. A common mistake is to always write a denominator of 2. 4 Write down both the solutions.
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Practice

5 Solve, giving your solutions in surd form.

a $3x^2 + 6x + 2 = 0$

b $2x^2 - 4x - 7 = 0$

6 Solve the equation $x^2 - 7x + 2 = 0$

Give your solutions in the form $\frac{a \pm \sqrt{b}}{c}$, where a , b and c are integers.

7 Solve $10x^2 + 3x + 3 = 5$

Give your solution in surd form.

Hint

Get all terms onto one side of the equation.

Extend

8 Choose an appropriate method to solve each quadratic equation, giving your answer in surd form when necessary.

a $4x(x - 1) = 3x - 2$

b $10 = (x + 1)^2$

c $x(3x - 1) = 10$

Answers

- 1**
- a** $x = 0$ or $x = -\frac{2}{3}$
- b** $x = 0$ or $x = \frac{3}{4}$
- c** $x = -5$ or $x = -2$
- d** $x = 2$ or $x = 3$
- e** $x = -1$ or $x = 4$
- f** $x = -5$ or $x = 2$
- g** $x = 4$ or $x = 6$
- h** $x = -6$ or $x = 6$
- i** $x = -7$ or $x = 4$
- j** $x = 3$
- k** $x = -\frac{1}{2}$ or $x = 4$
- l** $x = -\frac{2}{3}$ or $x = 5$
- 2**
- a** $x = -2$ or $x = 5$
- b** $x = -1$ or $x = 3$
- c** $x = -8$ or $x = 3$
- d** $x = -6$ or $x = 7$
- e** $x = -5$ or $x = 5$
- f** $x = -4$ or $x = 7$
- g** $x = -3$ or $x = 2\frac{1}{2}$
- h** $x = -\frac{1}{3}$ or $x = 2$
- 3**
- a** $x = 2 + \sqrt{7}$ or $x = 2 - \sqrt{7}$
- b** $x = 5 + \sqrt{21}$ or $x = 5 - \sqrt{21}$
- c** $x = -4 + \sqrt{21}$ or $x = -4 - \sqrt{21}$
- d** $x = 1 + \sqrt{7}$ or $x = 1 - \sqrt{7}$
- e** $x = -2 + \sqrt{6.5}$ or $x = -2 - \sqrt{6.5}$
- f** $x = \frac{-3 + \sqrt{89}}{10}$ or $x = \frac{-3 - \sqrt{89}}{10}$
- 4**
- a** $x = 1 + \sqrt{14}$ or $x = 1 - \sqrt{14}$
- b** $x = \frac{-3 + \sqrt{23}}{2}$ or $x = \frac{-3 - \sqrt{23}}{2}$
- c** $x = \frac{5 + \sqrt{13}}{2}$ or $x = \frac{5 - \sqrt{13}}{2}$
- 5**
- a** $x = -1 + \frac{\sqrt{3}}{3}$ or $x = -1 - \frac{\sqrt{3}}{3}$
- b** $x = 1 + \frac{3\sqrt{2}}{2}$ or $x = 1 - \frac{3\sqrt{2}}{2}$
- 6** $x = \frac{7 + \sqrt{41}}{2}$ or $x = \frac{7 - \sqrt{41}}{2}$
- 7** $x = \frac{-3 + \sqrt{89}}{20}$ or $x = \frac{-3 - \sqrt{89}}{20}$
- 8**
- a** $x = \frac{7 + \sqrt{17}}{8}$ or $x = \frac{7 - \sqrt{17}}{8}$
- b** $x = -1 + \sqrt{10}$ or $x = -1 - \sqrt{10}$
- c** $x = -1\frac{2}{3}$ or $x = 2$

Solving linear simultaneous equations using the elimination method

A LEVEL LINKS

Scheme of work: 1c. Equations – quadratic/linear simultaneous

Key points

- Two equations are simultaneous when they are both true at the same time.
- Solving simultaneous linear equations in two unknowns involves finding the value of each unknown which works for both equations.
- Make sure that the coefficient of one of the unknowns is the same in both equations.
- Eliminate this equal unknown by either subtracting or adding the two equations.

Examples

Example 1 Solve the simultaneous equations $3x + y = 5$ and $x + y = 1$

$\begin{array}{r} 3x + y = 5 \\ - \quad x + y = 1 \\ \hline 2x \quad = 4 \end{array}$ So $x = 2$ Using $x + y = 1$ $2 + y = 1$ So $y = -1$ Check: equation 1: $3 \times 2 + (-1) = 5$ YES equation 2: $2 + (-1) = 1$ YES	1 Subtract the second equation from the first equation to eliminate the y term. 2 To find the value of y, substitute $x = 2$ into one of the original equations. 3 Substitute the values of x and y into both equations to check your answers.
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Example 2 Solve $x + 2y = 13$ and $5x - 2y = 5$ simultaneously.

$\begin{array}{r} x + 2y = 13 \\ + \quad 5x - 2y = 5 \\ \hline 6x \quad = 18 \end{array}$ So $x = 3$ Using $x + 2y = 13$ $3 + 2y = 13$ So $y = 5$ Check: equation 1: $3 + 2 \times 5 = 13$ YES equation 2: $5 \times 3 - 2 \times 5 = 5$ YES	1 Add the two equations together to eliminate the y term. 2 To find the value of y, substitute $x = 3$ into one of the original equations. 3 Substitute the values of x and y into both equations to check your answers.
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Example 3 Solve $2x + 3y = 2$ and $5x + 4y = 12$ simultaneously.

$(2x + 3y = 2) \times 4 \rightarrow 8x + 12y = 8$ $(5x + 4y = 12) \times 3 \rightarrow \frac{15x + 12y = 36}{7x = 28}$ So $x = 4$ Using $2x + 3y = 2$ $2 \times 4 + 3y = 2$ So $y = -2$ Check: equation 1: $2 \times 4 + 3 \times (-2) = 2$ YES equation 2: $5 \times 4 + 4 \times (-2) = 12$ YES	1 Multiply the first equation by 4 and the second equation by 3 to make the coefficient of y the same for both equations. Then subtract the first equation from the second equation to eliminate the y term. 2 To find the value of y, substitute $x = 4$ into one of the original equations. 3 Substitute the values of x and y into both equations to check your answers.
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Practice

Solve these simultaneous equations.

1 $4x + y = 8$
 $x + y = 5$

2 $3x + y = 7$
 $3x + 2y = 5$

3 $4x + y = 3$
 $3x - y = 11$

4 $3x + 4y = 7$
 $x - 4y = 5$

5 $2x + y = 11$
 $x - 3y = 9$

6 $2x + 3y = 11$
 $3x + 2y = 4$

Solving linear simultaneous equations using the substitution method

A LEVEL LINKS

Scheme of work: 1c. Equations – quadratic/linear simultaneous

Textbook: Pure Year 1, 3.1 Linear simultaneous equations

Key points

- The substitution method is the method most commonly used for A level. This is because it is the method used to solve linear and quadratic simultaneous equations.

Examples

Example 4 Solve the simultaneous equations $y = 2x + 1$ and $5x + 3y = 14$

$5x + 3(2x + 1) = 14$ $5x + 6x + 3 = 14$ $11x + 3 = 14$ $11x = 11$ $\text{So } x = 1$ $\text{Using } y = 2x + 1$ $y = 2 \times 1 + 1$ $\text{So } y = 3$ Check: $\text{equation 1: } 3 = 2 \times 1 + 1 \quad \text{YES}$ $\text{equation 2: } 5 \times 1 + 3 \times 3 = 14 \quad \text{YES}$	1 Substitute $2x + 1$ for y into the second equation. 2 Expand the brackets and simplify. 3 Work out the value of x. 4 To find the value of y, substitute $x = 1$ into one of the original equations. 5 Substitute the values of x and y into both equations to check your answers.
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Example 5 Solve $2x - y = 16$ and $4x + 3y = -3$ simultaneously.

$y = 2x - 16$ $4x + 3(2x - 16) = -3$ $4x + 6x - 48 = -3$ $10x - 48 = -3$ $10x = 45$ $\text{So } x = 4\frac{1}{2}$ $\text{Using } y = 2x - 16$ $y = 2 \times 4\frac{1}{2} - 16$ $\text{So } y = -7$ Check: $\text{equation 1: } 2 \times 4\frac{1}{2} - (-7) = 16 \quad \text{YES}$ $\text{equation 2: } 4 \times 4\frac{1}{2} + 3 \times (-7) = -3 \quad \text{YES}$	1 Rearrange the first equation. 2 Substitute $2x - 16$ for y into the second equation. 3 Expand the brackets and simplify. 4 Work out the value of x. 5 To find the value of y, substitute $x = 4\frac{1}{2}$ into one of the original equations. 6 Substitute the values of x and y into both equations to check your answers.
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Practice

Solve these simultaneous equations.

7 $y = x - 4$
 $2x + 5y = 43$

8 $y = 2x - 3$
 $5x - 3y = 11$

9 $2y = 4x + 5$
 $9x + 5y = 22$

10 $2x = y - 2$
 $8x - 5y = -11$

11 $3x + 4y = 8$
 $2x - y = -13$

12 $3y = 4x - 7$
 $2y = 3x - 4$

13 $3x = y - 1$
 $2y - 2x = 3$

14 $3x + 2y + 1 = 0$
 $4y = 8 - x$

Extend

15 Solve the simultaneous equations $3x + 5y - 20 = 0$ and $2(x + y) = \frac{3(y - x)}{4}$.

Answers

1 $x = 1, y = 4$

2 $x = 3, y = -2$

3 $x = 2, y = -5$

4 $x = 3, y = -\frac{1}{2}$

5 $x = 6, y = -1$

6 $x = -2, y = 5$

7 $x = 9, y = 5$

8 $x = -2, y = -7$

9 $x = \frac{1}{2}, y = 3\frac{1}{2}$

10 $x = \frac{1}{2}, y = 3$

11 $x = -4, y = 5$

12 $x = -2, y = -5$

13 $x = \frac{1}{4}, y = 1\frac{3}{4}$

14 $x = -2, y = 2\frac{1}{2}$

15 $x = -2\frac{1}{2}, y = 5\frac{1}{2}$

Solving linear and quadratic simultaneous equations

A LEVEL LINKS

Scheme of work: 1c. Equations – quadratic/linear simultaneous

Key points

- Make one of the unknowns the subject of the linear equation (rearranging where necessary).
- Use the linear equation to substitute into the quadratic equation.
- There are usually two pairs of solutions.

Examples

Example 1 Solve the simultaneous equations $y = x + 1$ and $x^2 + y^2 = 13$

$x^2 + (x + 1)^2 = 13$ $x^2 + x^2 + x + x + 1 = 13$ $2x^2 + 2x + 1 = 13$ $2x^2 + 2x - 12 = 0$ $(2x - 4)(x + 3) = 0$ So $x = 2$ or $x = -3$ Using $y = x + 1$ When $x = 2$, $y = 2 + 1 = 3$ When $x = -3$, $y = -3 + 1 = -2$ So the solutions are $x = 2, y = 3$ and $x = -3, y = -2$ Check: equation 1: $3 = 2 + 1$ YES and $-2 = -3 + 1$ YES equation 2: $2^2 + 3^2 = 13$ YES and $(-3)^2 + (-2)^2 = 13$ YES	1 Substitute $x + 1$ for y into the second equation. 2 Expand the brackets and simplify. 3 Factorise the quadratic equation. 4 Work out the values of x. 5 To find the value of y, substitute both values of x into one of the original equations. 6 Substitute both pairs of values of x and y into both equations to check your answers.
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Example 2 Solve $2x + 3y = 5$ and $2y^2 + xy = 12$ simultaneously.

$x = \frac{5-3y}{2}$ $2y^2 + \left(\frac{5-3y}{2}\right)y = 12$ $2y^2 + \frac{5y-3y^2}{2} = 12$ $4y^2 + 5y - 3y^2 = 24$ $y^2 + 5y - 24 = 0$ $(y+8)(y-3) = 0$ So $y = -8$ or $y = 3$ Using $2x + 3y = 5$ When $y = -8$, $2x + 3 \times (-8) = 5$, $x = 14.5$ When $y = 3$, $2x + 3 \times 3 = 5$, $x = -2$ So the solutions are $x = 14.5$, $y = -8$ and $x = -2$, $y = 3$ Check: equation 1: $2 \times 14.5 + 3 \times (-8) = 5$ YES and $2 \times (-2) + 3 \times 3 = 5$ YES equation 2: $2 \times (-8)^2 + 14.5 \times (-8) = 12$ YES and $2 \times (3)^2 + (-2) \times 3 = 12$ YES	1 Rearrange the first equation. 2 Substitute $\frac{5-3y}{2}$ for x into the second equation. Notice how it is easier to substitute for x than for y. 3 Expand the brackets and simplify. 4 Factorise the quadratic equation. 5 Work out the values of y. 6 To find the value of x, substitute both values of y into one of the original equations. 7 Substitute both pairs of values of x and y into both equations to check your answers.
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Practice

Solve these simultaneous equations.

- | | |
|---|---|
| 1 $y = 2x + 1$
$x^2 + y^2 = 10$ | 2 $y = 6 - x$
$x^2 + y^2 = 20$ |
| 3 $y = x - 3$
$x^2 + y^2 = 5$ | 4 $y = 9 - 2x$
$x^2 + y^2 = 17$ |
| 5 $y = 3x - 5$
$y = x^2 - 2x + 1$ | 6 $y = x - 5$
$y = x^2 - 5x - 12$ |
| 7 $y = x + 5$
$x^2 + y^2 = 25$ | 8 $y = 2x - 1$
$x^2 + xy = 24$ |
| 9 $y = 2x$
$y^2 - xy = 8$ | 10 $2x + y = 11$
$xy = 15$ |

Extend

- | | |
|--|---|
| 11 $x - y = 1$
$x^2 + y^2 = 3$ | 12 $y - x = 2$
$x^2 + xy = 3$ |
|--|---|

Answers

1 $x = 1, y = 3$

$$x = -\frac{9}{5}, y = -\frac{13}{5}$$

2 $x = 2, y = 4$

$$x = 4, y = 2$$

3 $x = 1, y = -2$

$$x = 2, y = -1$$

4 $x = 4, y = 1$

$$x = \frac{16}{5}, y = \frac{13}{5}$$

5 $x = 3, y = 4$

$$x = 2, y = 1$$

6 $x = 7, y = 2$

$$x = -1, y = -6$$

7 $x = 0, y = 5$

$$x = -5, y = 0$$

8 $x = -\frac{8}{3}, y = -\frac{19}{3}$

$$x = 3, y = 5$$

9 $x = -2, y = -4$

$$x = 2, y = 4$$

10 $x = \frac{5}{2}, y = 6$

$$x = 3, y = 5$$

11 $x = \frac{1+\sqrt{5}}{2}, y = \frac{-1+\sqrt{5}}{2}$

$$x = \frac{1-\sqrt{5}}{2}, y = \frac{-1-\sqrt{5}}{2}$$

12 $x = \frac{-1+\sqrt{7}}{2}, y = \frac{3+\sqrt{7}}{2}$

$$x = \frac{-1-\sqrt{7}}{2}, y = \frac{3-\sqrt{7}}{2}$$

Linear inequalities

A LEVEL LINKS

Scheme of work: 1d. Inequalities – linear and quadratic (including graphical solutions)

Key points

- Solving linear inequalities uses similar methods to those for solving linear equations.
- When you multiply or divide an inequality by a negative number you need to reverse the inequality sign, e.g. $<$ becomes $>$.

Examples

Example 1 Solve $-8 \leq 4x < 16$

$\begin{aligned} -8 &\leq 4x < 16 \\ -2 &\leq x < 4 \end{aligned}$	Divide all three terms by 4.
--	------------------------------

Example 2 Solve $4 \leq 5x < 10$

$\begin{aligned} 4 &\leq 5x < 10 \\ \frac{4}{5} &\leq x < 2 \end{aligned}$	Divide all three terms by 5.
--	------------------------------

Example 3 Solve $2x - 5 < 7$

$\begin{aligned} 2x - 5 &< 7 \\ 2x &< 12 \\ x &< 6 \end{aligned}$	- Add 5 to both sides. - Divide both sides by 2.
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Example 4 Solve $2 - 5x \geq -8$

$\begin{aligned} 2 - 5x &\geq -8 \\ -5x &\geq -10 \\ x &\leq 2 \end{aligned}$	- Subtract 2 from both sides. - Divide both sides by -5. Remember to reverse the inequality when dividing by a negative number.
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Example 5 Solve $4(x - 2) > 3(9 - x)$

$\begin{aligned} 4(x - 2) &> 3(9 - x) \\ 4x - 8 &> 27 - 3x \\ 7x - 8 &> 27 \\ 7x &> 35 \\ x &> 5 \end{aligned}$	- Expand the brackets. - Add $3x$ to both sides. - Add 8 to both sides. - Divide both sides by 7.
---	---

Practice

1 Solve these inequalities.

a $4x > 16$

b $5x - 7 \leq 3$

c $1 \geq 3x + 4$

d $5 - 2x < 12$

e $\frac{x}{2} \geq 5$

f $8 < 3 - \frac{x}{3}$

2 Solve these inequalities.

a $\frac{x}{5} < -4$

b $10 \geq 2x + 3$

c $7 - 3x > -5$

3 Solve

a $2 - 4x \geq 18$

b $3 \leq 7x + 10 < 45$

c $6 - 2x \geq 4$

d $4x + 17 < 2 - x$

e $4 - 5x < -3x$

f $-4x \geq 24$

4 Solve these inequalities.

a $3t + 1 < t + 6$

b $2(3n - 1) \geq n + 5$

5 Solve.

a $3(2 - x) > 2(4 - x) + 4$

b $5(4 - x) > 3(5 - x) + 2$

Extend

6 Find the set of values of x for which $2x + 1 > 11$ and $4x - 2 > 16 - 2x$.

Answers

- | | | | |
|----------|---|-------------------------------|----------------------|
| 1 | a $x > 4$ | b $x \leq 2$ | c $x \leq -1$ |
| | d $x > -\frac{7}{2}$ | e $x \geq 10$ | f $x < -15$ |
| 2 | a $x < -20$ | b $x \leq 3.5$ | c $x < 4$ |
| 3 | a $x \leq -4$ | b $-1 \leq x < 5$ | c $x \leq 1$ |
| | d $x < -3$ | e $x > 2$ | f $x \leq -6$ |
| 4 | a $t < \frac{5}{2}$ | b $n \geq \frac{7}{5}$ | |
| 5 | a $x < -6$ | b $x < \frac{3}{2}$ | |
| 6 | $x > 5$ (which also satisfies $x > 3$) | | |

Quadratic inequalities

A LEVEL LINKS

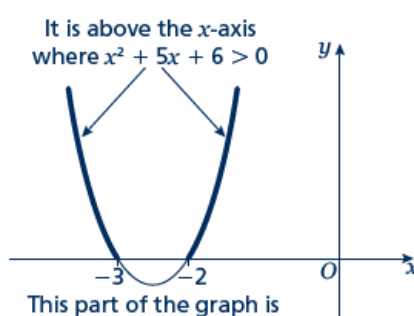
Scheme of work: 1d. Inequalities – linear and quadratic (including graphical solutions)

Key points

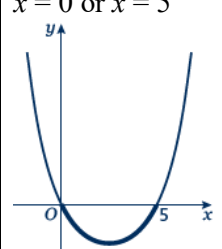
- First replace the inequality sign by = and solve the quadratic equation.
- Sketch the graph of the quadratic function.
- Use the graph to find the values which satisfy the quadratic inequality.

Examples

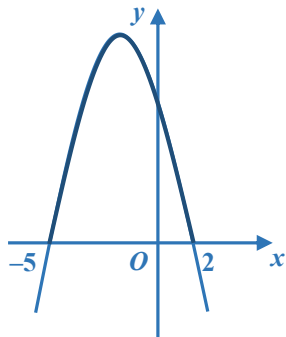
Example 1 Find the set of values of x which satisfy $x^2 + 5x + 6 > 0$

$x^2 + 5x + 6 = 0$ $(x + 3)(x + 2) = 0$ $x = -3 \text{ or } x = -2$  $x < -3 \text{ or } x > -2$	1 Solve the quadratic equation by factorising. 2 Sketch the graph of $y = (x + 3)(x + 2)$ 3 Identify on the graph where $x^2 + 5x + 6 > 0$, i.e. where $y > 0$ 4 Write down the values which satisfy the inequality $x^2 + 5x + 6 > 0$
--	--

Example 2 Find the set of values of x which satisfy $x^2 - 5x \leq 0$

$x^2 - 5x = 0$ $x(x - 5) = 0$ $x = 0 \text{ or } x = 5$  $0 \leq x \leq 5$	1 Solve the quadratic equation by factorising. 2 Sketch the graph of $y = x(x - 5)$ 3 Identify on the graph where $x^2 - 5x \leq 0$, i.e. where $y \leq 0$ 4 Write down the values which satisfy the inequality $x^2 - 5x \leq 0$
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Example 3 Find the set of values of x which satisfy $-x^2 - 3x + 10 \geq 0$

$-x^2 - 3x + 10 = 0$ $(-x + 2)(x + 5) = 0$ $x = 2 \text{ or } x = -5$  $-5 \leq x \leq 2$	1 Solve the quadratic equation by factorising. 2 Sketch the graph of $y = (-x + 2)(x + 5) = 0$ 3 Identify on the graph where $-x^2 - 3x + 10 \geq 0$, i.e. where $y \geq 0$ 3 Write down the values which satisfy the inequality $-x^2 - 3x + 10 \geq 0$
--	---

Practice

- 1** Find the set of values of x for which $(x + 7)(x - 4) \leq 0$
- 2** Find the set of values of x for which $x^2 - 4x - 12 \geq 0$
- 3** Find the set of values of x for which $2x^2 - 7x + 3 < 0$
- 4** Find the set of values of x for which $4x^2 + 4x - 3 > 0$
- 5** Find the set of values of x for which $12 + x - x^2 \geq 0$

Extend

Find the set of values which satisfy the following inequalities.

- 6** $x^2 + x \leq 6$
- 7** $x(2x - 9) < -10$
- 8** $6x^2 \geq 15 + x$

Answers

1 $-7 \leq x \leq 4$

2 $x \leq -2$ or $x \geq 6$

3 $\frac{1}{2} < x < 3$

4 $x < -\frac{3}{2}$ or $x > \frac{1}{2}$

5 $-3 \leq x \leq 4$

6 $-3 \leq x \leq 2$

7 $2 < x < 2\frac{1}{2}$

8 $x \leq -\frac{3}{2}$ or $x \geq \frac{5}{3}$

Rearranging equations

A LEVEL LINKS

Scheme of work: 6a. Definition, differentiating polynomials, second derivatives

Textbook: Pure Year 1, 12.1 Gradients of curves

Key points

- To change the subject of a formula, get the terms containing the subject on one side and everything else on the other side.
- You may need to factorise the terms containing the new subject.

Examples

Example 1 Make t the subject of the formula $v = u + at$.

$v = u + at$ $v - u = at$ $t = \frac{v - u}{a}$	1 Get the terms containing t on one side and everything else on the other side. 2 Divide throughout by a.
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Example 2 Make t the subject of the formula $r = 2t - \pi t$.

$r = 2t - \pi t$ $r = t(2 - \pi)$ $t = \frac{r}{2 - \pi}$	1 All the terms containing t are already on one side and everything else is on the other side. 2 Factorise as t is a common factor. 3 Divide throughout by $2 - \pi$.
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Example 3 Make t the subject of the formula $\frac{t + r}{5} = \frac{3t}{2}$.

$\frac{t + r}{5} = \frac{3t}{2}$ $2t + 2r = 15t$ $2r = 13t$ $t = \frac{2r}{13}$	1 Remove the fractions first by multiplying throughout by 10. 2 Get the terms containing t on one side and everything else on the other side and simplify. 3 Divide throughout by 13.
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Example 4 Make t the subject of the formula $r = \frac{3t+5}{t-1}$.

$r = \frac{3t+5}{t-1}$ $r(t-1) = 3t+5$ $rt - r = 3t+5$ $rt - 3t = 5 + r$ $t(r-3) = 5 + r$ $t = \frac{5+r}{r-3}$	1 Remove the fraction first by multiplying throughout by $t-1$. 2 Expand the brackets. 3 Get the terms containing t on one side and everything else on the other side. 4 Factorise the LHS as t is a common factor. 5 Divide throughout by $r-3$.
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Practice

Change the subject of each formula to the letter given in the brackets.

- | | | |
|---|-----------------------------------|-----------------------------------|
| 1 $C = \pi d$ [d] | 2 $P = 2l + 2w$ [w] | 3 $D = \frac{S}{T}$ [T] |
| 4 $p = \frac{q-r}{t}$ [t] | 5 $u = at - \frac{1}{2}t$ [t] | 6 $V = ax + 4x$ [x] |
| 7 $\frac{y-7x}{2} = \frac{7-2y}{3}$ [y] | 8 $x = \frac{2a-1}{3-a}$ [a] | 9 $x = \frac{b-c}{d}$ [d] |
| 10 $h = \frac{7g-9}{2+g}$ [g] | 11 $e(9+x) = 2e+1$ [e] | 12 $y = \frac{2x+3}{4-x}$ [x] |

13 Make r the subject of the following formulae.

- | | | | |
|-----------------|----------------------------|--------------------|------------------------------|
| a $A = \pi r^2$ | b $V = \frac{4}{3}\pi r^3$ | c $P = \pi r + 2r$ | d $V = \frac{2}{3}\pi r^2 h$ |
|-----------------|----------------------------|--------------------|------------------------------|

14 Make x the subject of the following formulae.

- | | |
|----------------------------------|---|
| a $\frac{xy}{z} = \frac{ab}{cd}$ | b $\frac{4\pi cx}{d} = \frac{3z}{py^2}$ |
|----------------------------------|---|

15 Make $\sin B$ the subject of the formula $\frac{a}{\sin A} = \frac{b}{\sin B}$

16 Make $\cos B$ the subject of the formula $b^2 = a^2 + c^2 - 2ac \cos B$.

Extend

17 Make x the subject of the following equations.

- | | |
|-----------------------------|--|
| a $\frac{p}{q}(sx+t) = x-1$ | b $\frac{p}{q}(ax+2y) = \frac{3p}{q^2}(x-y)$ |
|-----------------------------|--|

Answers

$$1 \quad d = \frac{C}{\pi}$$

$$2 \quad w = \frac{P-2l}{2}$$

$$3 \quad T = \frac{S}{D}$$

$$4 \quad t = \frac{q-r}{p}$$

$$5 \quad t = \frac{2u}{2a-1}$$

$$6 \quad x = \frac{V}{a+4}$$

$$7 \quad y = 2 + 3x$$

$$8 \quad a = \frac{3x+1}{x+2}$$

$$9 \quad d = \frac{b-c}{x}$$

$$10 \quad g = \frac{2h+9}{7-h}$$

$$11 \quad e = \frac{1}{x+7}$$

$$12 \quad x = \frac{4y-3}{2+y}$$

$$13 \quad \mathbf{a} \quad r = \sqrt{\frac{A}{\pi}}$$

$$\mathbf{b} \quad r = \sqrt[3]{\frac{3V}{4\pi}}$$

$$\mathbf{c} \quad r = \frac{P}{\pi+2}$$

$$\mathbf{d} \quad r = \sqrt{\frac{3V}{2\pi h}}$$

$$14 \quad \mathbf{a} \quad x = \frac{abz}{cdy}$$

$$\mathbf{b} \quad x = \frac{3dz}{4\pi cpy^2}$$

$$15 \quad \sin B = \frac{b \sin A}{a}$$

$$16 \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$17 \quad \mathbf{a} \quad x = \frac{q+pt}{q-ps}$$

$$\mathbf{b} \quad x = \frac{3py+2pqy}{3p-apq} = \frac{y(3+2q)}{3-aq}$$